# HW. # 11

Homework problems are taken from several textbooks. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Calculate the gradient.

1. $f(x,y) = \frac{3}{4}x^2y^4 - \frac{5}{4}xy^6$

2. $g(x,y) = x\sin(y) + y\cos^2(x)$

3. $h(x,y,z) = xe^{yz}$

4. $H(x_1,\ldots,x_n) = \ln\left(\sum_{i=1}^{n} x_i^2\right)$

5. $P(x_1,\ldots,x_n) = e^{\sum_{i=1}^{n} x_i^2}$

6. $f(x) = \|x\|^2$, $x \in R^n$

Calculate the directional derivative of f at the indicated point in the direction given. [Hint: Be careful!]

7. $f(x,y) = 7x^2 + 4xy - 3y^2$; Point **a** = $\left(\frac{2}{3}, \frac{3}{2}\right)$; in the direction of **v** = (1, 2)

8. $f(x,y) = x^{2/3} + y^{2/3}$; Point **a** = (1, 2); in the direction of **v** = (-3, 5)

9. $f(x,y) = \ln\left(x + e^y\right)$; Point **a** = (e, 1); in the direction of **v** = (e, 1)

10. $f(x,y,z) = \frac{\sqrt{x^2 + y^2}}{z}$; Point **a** = (3, -4, 1); directional vector points from **a** toward the origin.

11. $f(x, y, z) = \ln(xyz + 1)$; Point **a** = $\left(2, e, \frac{1}{e}\right)$; directional vector points from **a** toward (3, 6, e)

12. Captain Ralph is in trouble near the sunny side of Mercury. The temperature of the ship's hull when he is at location (x, y, z) will be given by T(x, y, z) = $e^{-x^2-2y^2-3z^2}$, where x, y, and z are measured in meters. He is currently at (1, 1, 1).

a) In what direction should he proceed in order to decrease the temperature most rapidly?
b) If the ship travels at $e^8$ meters per second, how fast will be the temperature decrease if he proceeds in that direction?
c) Unfortunately, the metal of the hull will crack if cooled at a rate greater than $\sqrt{14}e^2$ degrees per second. Describe the set of possible directions in which he may proceed to bring the temperature down at no more than that rate.

13. A bug finds itself in a toxic environment. The toxicity level is given by T(x, y) = $2x^2 - 4y^2$. The bug is at (-1, 2). In what direction should it move to lower the toxicity the fastest?

14. An engineer wishes to built a railroad up the mountain that has the shape of an elliptic paraboloid $z = c - ax^2 - by^2$, where a, b, and c are positive constants, x and y are the east-west and north-south map coordinates, and z is the altitude above sea level. Straight up the mountain is much too steep for the power of the engines. At the point (1, 1), in what directions may the track be laid so that it will be climbing with 3% grade—that is, an angle whose tangent is 0.03? (There are two possibilities).

15. Suppose that a particle is ejected from the surface $x^2 + y^2 - z^2 = -1$ at the point (1, 1, $\sqrt{3}$) along the normal directed toward the xy plane to the surface at time t = 0 with a speed of 10 units per second. When and where does it cross the xy plane?

16. In an earlier lecture, we have seen that a function may have all of its partial derivatives defined at a point and yet fail to be differentiable. Must a function be differentiable at a point if all the directional derivatives are defined at that point? Consider the function

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2} & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases}$$

Show that

a) $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (0, 0).

b) Show that f is not differentiable at (0, 0).

c) Show that if **u** = (a, b) is a direction vector, $D_u f(0,0)$ is defined and is equal to $\frac{ab^2}{a^2+b^2} = ab^2$, but $\nabla f(0,0) \bullet u = 0$.